INVESTIGATION OF THE STATISTICAL METHOD OF TIME DELAY ESTIMATION BASED ON CONDITIONAL AVERAGING OF DELAYED SIGNAL

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Abstract

This paper presents the results of the theoretical and practical analysis of selected features of the function of conditional average value of the absolute value of delayed signal (*CAAV*). The results obtained with the *CAAV* method have been compared with the results obtained by method of cross correlation (*CCF*), which is often used at the measurements of random signal time delay. The paper is divided into five sections. The first is devoted to a short introduction to the subject of the paper. The model of measured stochastic signals is described in Section 2. The fundamentals of time delay estimation using *CCF* and *CAAV* are presented in Section 3. The standard deviations of both functions in their extreme points are evaluated and compared. The results of experimental investigations are discussed in Section 4. Computer simulations were used to evaluate the performance of the *CAAV* and *CCF* methods. The signal and the noise were Gaussian random variables, produced by a pseudorandom noise generator. The experimental standard deviations of both functions for the chosen signal to noise ratio (*SNR*) were obtained and compared. All simulation results were averaged for 1000 independent runs. It should be noted that the experimental results were close to the theoretical values. The conclusions and final remarks were included in Section 5. The authors conclude that the *CAAV* method described in this paper has less standard deviation in the extreme point than *CCF* and can be applied to time delay measurement of random signals.

Keywords: time delay estimation, random signals, conditional averaging, cross-correlation.

1. Introduction

Time delay estimation is a problem quite frequently studied in signal processing. The problem is significant in such areas as radar technology, radio-astronomy, location of interference transfer paths or contact-free measurements of transport parameters. Determination of the time delay of stochastic signals received from two or more sensors is commonly carried out with the use of statistical methods. This problem has been thoroughly presented in the literature [1-8] which describes a variety of methods consisting of the analysis of signals in the time and frequency domains. Among the traditional methods used for stationary signals, the most common one is direct cross-correlation (*CCF*) in the time domain and the phase of cross-spectral power density in the frequency domain [3, 9-12]. Other approaches can be used in specific conditions: differential methods [4, 6], the correlation method with the Hilbert transform [1, 13-15] or relatively unpopular methods based on conditional averaging of signals [16-19].

This paper presents the results of selected research into the features of the method which uses the function of conditional averaging of the delayed signal absolute value (*CAAV*) [16, 17]. Theoretical and experimental standard deviations of both functions in extreme points were evaluated and compared with a discrete estimator of *CAAV* and a direct discrete estimator of *CCF* for the assumed signal models. The signal to noise ratio values (*SNR*) were determined for the assumed signal models, where the analyzed method is characterized by smaller standard deviations of estimation for specific parameters of the analysis.

2. Measurement signal models

In the case of many delay time estimations (i.e. measurements of transport parameters of solids and flows), the relation for signals $x(t)$ and $y(t)$ received from two sensors is usually given by the following formula [2]:

$$y(t) = c \cdot x(t - \tau_0) + z(t), \tag{1}$$

where: $x(t)$ is the stationary random signal with a normal probability distribution $N(0, \sigma_x)$, frequency band B and the spectral power density:

$$G_{xx}(f) = \begin{cases} G & 0 < f \leq B \\ 0 & f > B \end{cases}, \tag{2}$$

c, G are the constant factors; $\tau_0 = d/V$ is the transport delay equal to the quotient of the sensor sparing distance d and the average velocity of object V; $z(t)$ is the white noise, non-correlated with signal $x(t)$, with the distribution of $N(0, \sigma_z)$. The autocorrelation function of signal $x(t)$ has the following form:

$$R_{xx}(\tau) = GB \left(\frac{\sin 2\pi B \tau}{2\pi B \tau} \right). \tag{3}$$

3. Direct cross-correlation and the arithmetic conditional average value of the delayed signal absolute value in the time delay estimation

Cross-correlation $R_{xy}(\tau)$ of signals described with the relation (1) can be given in the following form [1]:

$$R_{xy}(\tau) = E[(x(t)y(t + \tau)] = c R_{xx}(\tau - \tau_0), \tag{4}$$

where $E[\ ]$ is the expected value operation. The function (4) achieves the maximum value for $\tau = \tau_0$, so that the delay can be determined as the argument of the main extreme of this function:

$$\tau_0 = arg\{max R_{xy}(\tau)\} = arg\{R_{xy}(\tau_0)\}. \tag{5}$$

Following normalization, the correlation (4) is equal to:

$$\rho_{xy}(\tau) = \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} = \frac{cR_{xx}(\tau - \tau_0)}{\sigma_x \sigma_y}, \tag{6}$$

while when replacing with $\tau = \tau_0$:

$$\rho_{xy}(\tau_0) = \frac{R_{xx}(0)}{\sigma_x \sigma_y} = \frac{c\sigma_x^2}{\sigma_x \sigma_y}. \tag{7}$$

Since signals $x(t)$ and $z(t)$ are non-correlated:

$$\sigma_y = \sqrt{(c\sigma_x)^2 + \sigma_z^2}. \tag{8}$$

When replacing (7) with (8) and defining the signal to noise ratio as $SNR = (\sigma_x/\sigma_z)^2$, the result is:

$$\rho_{xy}(\tau_0) = \left[1 + \frac{1}{c^2 SNR} \right]^{-1/2}. \tag{9}$$

When there is no disturbance $\rho_{xy}(\tau_0) = 1$.

If signals $x(t)$ and $y(t)$ have the lengths T_{total}, the standard deviation of the *CCF* estimator can be formulated as follows [1]:

$$\sigma\left[\hat{R}_{xy}(\tau_0)\right] \approx \left\{\frac{1}{2BT_{total}}\left[R_{xx}(0)R_{yy}(0)+R_{xy}^2(\tau_0)\right]\right\}^{1/2}. \tag{10}$$

The relation (10) is just for the high values of T_{total} ($T_{total} \geq 10|\tau|$ and $BT_{total} \geq 5$).

As for the computer methods of analysis, $2BT_{total} = N_{total}$ [2, 13] is assumed, where $N_{total} = T_{total} / \Delta t$, and Δt is the properly selected sampling interval. When estimating the *CCF* with the pairs of non-correlated samples divided into N cycles and by transforming (10) taking (9) into consideration, the result is the relation of the normalized standard deviation ε:

$$\varepsilon\left[\hat{R}_{xy}(\tau_0)\right] = \frac{\sigma\left[\hat{R}_{xy}(\tau_0)\right]}{R_{xy}(\tau_0)} \approx \left[\frac{1}{N}\left(2+\frac{1}{c^2 SNR}\right)\right]^{1/2}. \tag{11}$$

The discrete *CCF* estimator can be expressed as:

$$\hat{R}_{xy}(l) = \frac{1}{N-l}\sum_{n=1}^{N-l} x(n)y(n+l), \tag{12}$$

where: $l = \tau/\Delta t$, $n = t/\Delta t$.

To obtain of time delay τ_0 it is possible to use also terms: a minimum of conditional variance $\sigma_{y|x}^2(\tau)$ or a minimum of conditional expected value $A_{|y|\|x=0}$. The expected conditional value of the delayed signal absolute value for the condition $x(t) = 0$ (the formula is simplified as follows: $y(t) = y$ and $x(t) = x$) is defined as follows [16]:

$$A_{|y|} = A_{|y|\|x=0} = E\left\{|y|\big\|_{x=0}\right\} = \int_0^\infty |y| \cdot p\left(|y|\big\|_{x=0}\right) dy, \tag{13}$$

where $p\left(|y|\big\|_{x=0}\right)$ is the conditional probability density for the signal absolute value y at the condition $x = 0$.

A good estimator of the expected conditional value (13) is the arithmetic conditional average value of the delayed signal absolute value (*CAAV*). In practice, its determination entails detection of the non-cross-correlated instant of zero transition of the original signal $x(t)$, starting the registration of the delayed signal $y(t)$ fragments in those moments and averaging the set of their absolute value. The discrete estimator of *CAAV* can be formulated as follows [18]:

$$\hat{A}_{|y|}(l) = \overline{|y(l)|}\big\|_{x(n)=0} = \frac{1}{M}\sum_{n=1}^{M}|y(n+l)|\big\|_{x(n)=0}, \tag{14}$$

where M is the number of zero transitions of the original signal included during the determination of *CAAV*.

The relationship of *CAAV* and the normalized *CCF* is defined by the formula [16]:

$$A_{|y|}(\tau) = \sigma_y \sqrt{\frac{2}{\pi}\left[1-\rho_{xy}^2(\tau)\right]}, \tag{15}$$

and the following relation applied for the normalized *CAAV* $a_{|y|}(\tau)$:

$$a_{|y|}(\tau) = \frac{A_{|y|}(\tau)}{A_{|y|}(\tau)_{max}} = \sqrt{1-\rho_{xy}^2(\tau)}. \tag{16}$$

The transport delay can be determined as the argument of the main minimum of the function (15) or (16):

$$\tau_0 = arg\{min\, A_{|y|}(\tau)\} = arg\,\{\,A_{|y|}(\tau_0)\}. \tag{17}$$

Examples of runs of the normalized functions *CCF* and *CAAV* for $z(t) = 0$ are given in Fig. 1.
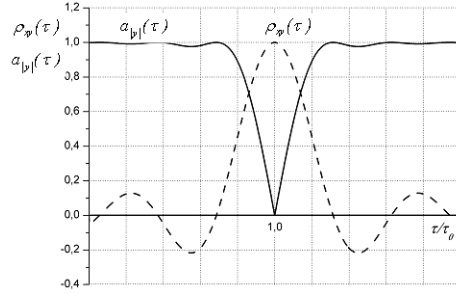

Fig. 1. Examples of normalized *CCF* and *CAAV* functions.

An increase of the disturbance value results in the decrease in the main *CCF* maximum in lieu with the relation (9) and, correspondingly, an increase of the main minimum of *CAAV*.

The relative standard deviation of the *CAAV* estimator, where $\tau = \tau_0$, can be formulated as follows [18]:

$$\varepsilon\big[\hat{A}_{|y|}(\tau_0)\big] = \frac{\sigma\big[\hat{A}_{|y|}(\tau_0)\big]}{A_{|y|}(\tau)_{max}} \approx \left[\frac{1}{M}\left(\frac{\pi}{2}-1\right)\frac{1}{1+c^2 SNR}\right]^{1/2}. \tag{18}$$

The comparison of (18) and (11) results in the following:

$$\frac{\varepsilon\big[\hat{A}_{|y|}(\tau_0)\big]}{\varepsilon\big[\hat{R}_{xy}(\tau_0)\big]} \approx \left[\frac{N}{M}\frac{\left(\frac{\pi}{2}-1\right)\left(\frac{1}{1+c^2 SNR}\right)}{\left(2+\frac{1}{c^2 SNR}\right)}\right]^{1/2}. \tag{19}$$

The plot of relation (19) where the factor $c = 1$, defining several values of the *N/M* ratio, is given in Fig. 2. The analysed *SNR* range shows that the relative standard deviation of *CAAV* is always less than the corresponding deviation of *CCF* when the *N/M* ration is equal to or less than 10. In reality, the *N/M* ratio value depends on the correlation interval of the measurement signals, which determine the choice of non-correlated samples.
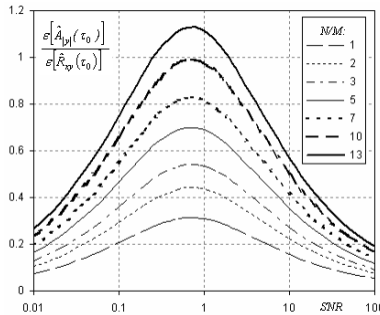

Fig. 2. Plots of the relation (19) for selected values of *N/M* ratio.

4. Results of simulations

For verification of theoretical considerations the practical analysis was conducted using an application in the LabVIEW software environment. Using the computer simulation it is possible to examination of influence of experimental parameters for *CAAV* and *CCF* characteristics. Reciprocally delayed stochastic signals were generated, which corresponded to the model (1), followed by the determination of discrete *CCF* (12) and *CAAV* (14) estimators for the given *SNR* values, taking into account the non-correlated sample pairs.

The assumed number of samples was 200,000, the standard signal deviation $\sigma_x = 1$, $c = 1$ and the transport delay l_0 was 100 samples. The examples of runs of the obtained *CCF* and *CAAV* characteristics for $\sigma_z = 0$ have been presented in Fig. 3. The next stage of the practical analysis entailed multiple repetitions of the simulation and determination of the relative experimental standard deviations for the determined characteristics at the extreme points:

$$\hat{\varepsilon}\left[\hat{R}_{xy}(l)\right] = \left[\frac{1}{K}\sum_{i=1}^{K}\left\{\frac{\hat{R}_{xy}(l)_i - \overline{\hat{R}_{xy}(l)}}{\left[\hat{R}_{xy}(l)_i\right]_{max}}\right\}^2\right]^{1/2} ; \tag{20}$$

$$\hat{\varepsilon}\left[\hat{A}_{|y|}(l)\right] = \left[\frac{1}{K}\sum_{i=1}^{K}\left\{\frac{\hat{A}_{|y|}(l)_i - \overline{\hat{A}_{|y|}(l)}}{\left[\hat{A}_{|y|}(l)_i\right]_{max}}\right\}^2\right]^{1/2} , \tag{21}$$

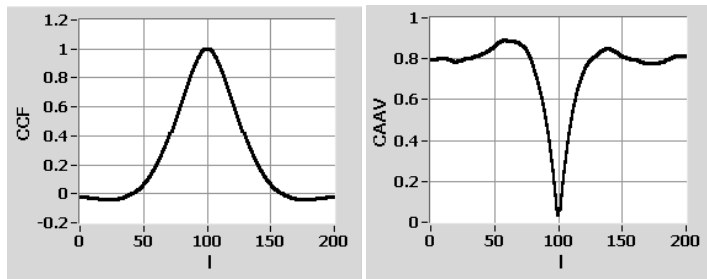where: *K* is the number of test repetitions.

Fig. 3. *CCF* and *CAAV* obtained from the simulation.

The simulations were run for the given *SNR* values. Fig. 4 and 5 show example runs of *CCF* and *CAAV* at extreme points, where the single relative standard deviation range is shown as determined using the formulas (20) and (21), for $K = 1000$. The characteristics shown in Fig. 4 are obtained for $\sigma_z = 0$.
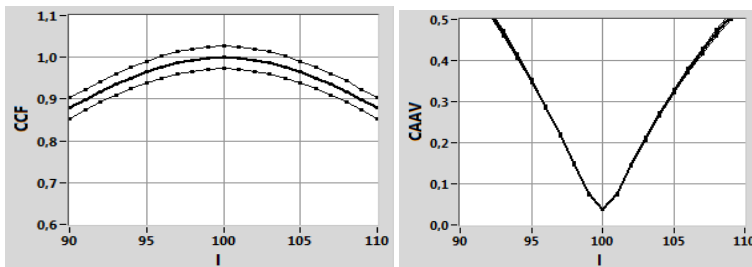

Fig. 4. *CCF* and *CAAV* for $\sigma_z = 0$.

The runs of *CCF* and *CAAV* obtained for *SNR* = 4 (a), *SNR* = 1 (b) and *SNR* = 0.25 (c) accordingly where the given *N/M* = 2 for the modelled signals, are shown in Fig. 5.
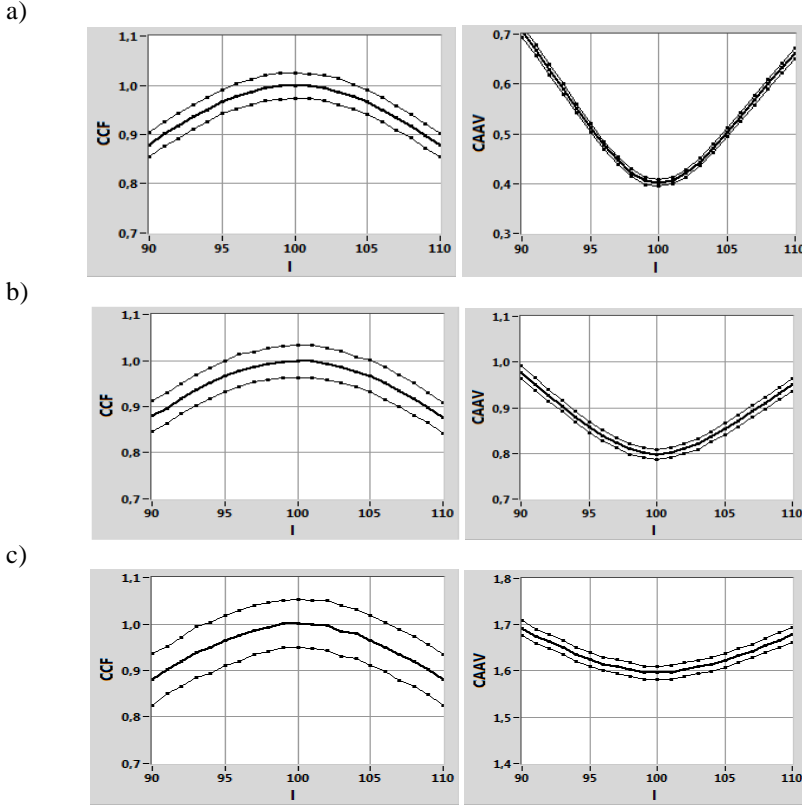
a)

b)

c)

Fig. 5. *CCF* and *CAAV* for *SNR* = 4 (a), *SNR* = 1 (b) and *SNR* = 0.25 (c).

By determining the quotient of the relations (21) and (20), the relative standard deviations of *CAAV* and *CCF* can be compared for the selected delay depending on the *SNR*. Fig. 6 presents the plot of the relations:

$$\hat{\varepsilon}\big[\hat{A}_{|y|}(l_0)\big] \big/ \hat{\varepsilon}\big[\hat{R}_{xy}(l_0)\big] = f(SNR), \tag{22}$$

compared to the theoretical run (19) for *N/M* = 1 and *N/M* = 2.
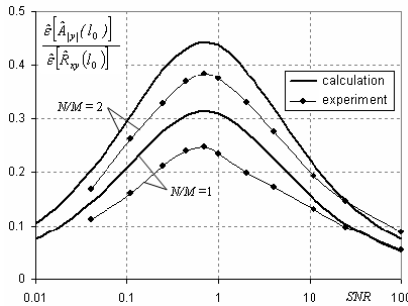

Fig. 6. Plots of relations (19) and (22) for *N/M* = 1 and *N/M* =2.

The results of the practical analysis in the entire surveyed range of *SNR* do not depart greatly from the calculations (in favour of the *CAAV* characteristic), which can be specifically seen in the range of *SNR* values approximated to unity.

5. Conclusion

This work entailed the comparison of relative standard deviations of the direct cross-correlation to the conditional average value of the delayed signal absolute value at the extreme points for the assumed signal models and the given *SNR* values. The theoretical analysis implies that the relative standard deviation of the *CAAV* values in the range of *N/M* equal to or less than 10 is always less than the corresponding *CCF* values irrespective of the *SNR* values. The computer simulations confirm the results of theoretical deliberations and show an influence of experimental parameters for *CAAV* and *CCF* characteristics. The standard deviation values of *CCF* and *CAAV* at the extreme points significantly affect the uncertainty of the transport delay determined by using the functions. This problem and metrological properties of another non-linear extreme characteristics are currently undergoing further investigation.

Acknowledgements

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